# Sadisticube – Analysis of a New Puzzle

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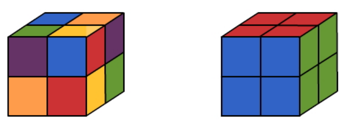
Mathematicians enjoy all sorts of puzzles – puzzles of knowledge, such as crosswords, puzzles of logic, such as Sudoku, jigsaw, which require an eye for shapes, metal tavern puzzles, which take a combination of manual and visual dexterity. Aside from the standard challenge, puzzles yield an extra level of entertainment – they can be analyzed mathematically, often providing insight into other applications. Puzzles that have been analyzed extensively include Rubik's cube, the Tower of Hanoi, and Sudoku. In this paper we will analyze Sadisticube, a simple block puzzle that turns out to be quite a bit trickier than it looks.

**The Puzzle**

Sadisticube consists of 8 individual blocks with colored faces. The blocks are to be assembled into a cube with each face consisting of four squares of the same color.



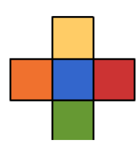
**No.** **Yes!**

The configuration of a solution does resemble the Rubik's cube; however, the mathematical analysis is quite different. At first glance, it seems very straightforward: reposition and manipulate the blocks until the colors all match up. People can occasionally solve this by patient effort and a bit of luck, or by dogged persistence (and a bit of luck). Without analysis, there is an element of luck involved, because of the nature of the solution and the number of combinations involved.

**The Blocks**

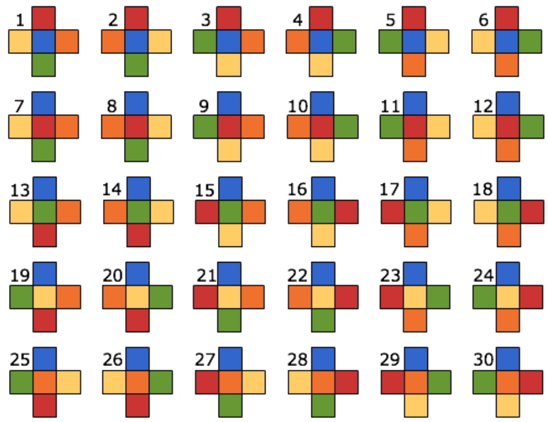
The eight blocks can be colored in a variety of ways. The standard puzzle, which we will analyze here, has each block colored with each face one of six different colors (red, green, blue, yellow, orange, and purple) and no colors repeated.

The easiest planar way to represent a block is like this:



Picture the flaps folding down, and the bottom colored purple.

There are thirty unique ways to color such a block:




**The Solution Cube**

Given a set of eight blocks, there are $1.85\times10^{14}$ unique ways of arranging them into a cube. There are 24 different ways of orienting each block – six faces to put on the bottom, and four sides to orient. This gives $24^8$ ways to arrange the eight blocks. The blocks can be assembled into a cube in 8! ways. However, the final cube can be oriented in 24 ways, we divide by this to get $24^7\cdot 8!\approx 1.85\times10^{14}$.

Depending how the blocks are painted, a set can have anywhere from no possible solutions (which would really be sadistic) up to half a dozen solutions. A solution is defined as the coloration of the final $2\times2\times2$ cube and can look like any of the 30 that the blocks themselves can have. It is also possible to have ‘improper’ solutions, in which a single color appears on more than one face.

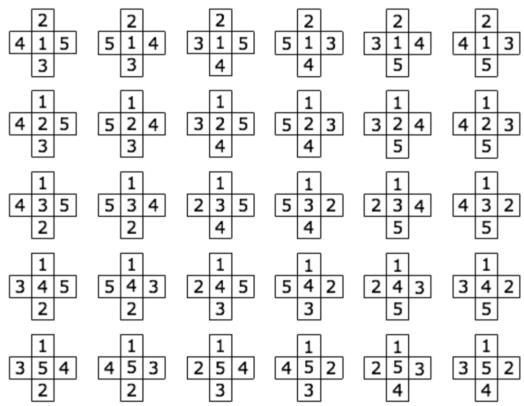
Because there are 30 different possible colorations for the solution, it is impossible to tell without some analysis which of them will work for a given set of blocks. If you are trying to construct a cube with red opposite yellow, for example, and this isn’t one of the solutions of your set, then you can manipulate the blocks ‘til the cows come home and the colors simply won’t match up. Hence the name.

The solution of a set can be found by analyzing the 8 blocks and determining which of the 30 cubes can be constructed from this set. This does not completely alleviate confusion, but “block” refers to one of the 8 individual blocks in the set, and “cube” refers to the $2\times2\times2$ possible solution.

## Finding a Solution

A Sadisticube puzzle is guaranteed to have at least one solution. The key to the analysis is to examine the vertices of the blocks. The cube has eight vertices and each one is a vertex of one block. If we can determine which solutions are possible, given the vertices we have for each of the eight blocks, we will at least know what to aim for.

To continue, we need to translate the faces from colors to numbers.

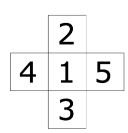



NOTE: This is arbitrary, but here we have

blue = 1   red = 2   green = 3   yellow = 4   orange = 5   purple = 6

A vertex is determined by its three adjacent faces, with the numbers assigned in *clockwise* order. Reading across the top from left to right and down the rows, we will refer to these as blocks 1 – 30.

For example, block 1 is




and has vertices: {125, 153, 134, 142, 652, 635, 643, 624}.

NOTE:  Vertex 125 is equivalent to 251 and 512. To avoid confusion, if a vertex contains a 6, it starts with the 6.  All others start with the smallest of the three digits.

The eight blocks in the set are compared one by one with each of the thirty possible solution cubes.

Let's say we have a set consisting of the following blocks: {3, 9, 12, 14, 17, 22, 26, 30}

We need to compare each block in our set vertex by vertex, with each of the thirty possible solution cubes.

| | |
|---|---|
| Cube 1 | {125, 153, 134, 142, 652, 635, 643, 624} |
| 1st Block – Block 3 | {125, 154, 143, 132, 652, 645, 634, 623} |

These blocks have vertices 125 and 652 in common. This means block 3 could fill either vertex 1 or vertex 5 of cube 1.

The second block has a problem.

| | |
|---|---|
| Cube 1 | {125, 153, 134, 142, 652, 635, 643, 624} |
| 2nd Block – Block 9 | {152, 254, 243, 123, 651, 645, 634, 613} |

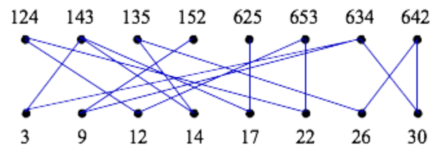
It has *no* vertices in common with cube 1. This means that block 9 could not fill any vertex of cube 1. Thus cube 1 cannot be a solution to this set.

Since this set of blocks is inconsistent with solution cube 1, we move onto cube 2.

| | |
|---|---|
| Cube 2 | {124, 143, 135, 152, 642, 634, 653, 625} |
| | |
| Block 3 | {125, 154, 143, 132, 652, 645, 634, 623} |
| Block 9 | {152, 254, 243, 123, 651, 645, 634, 613} |
| Block 12 | {132, 235, 254, 124, 631, 653, 645, 614} |
| Block 14 | {143, 234, 253, 135, 641, 624, 652, 615} |
| Block 17 | {143, 345, 235, 132, 641, 654, 625, 612} |
| Block 22 | {124, 234, 354, 145, 621, 632, 653, 615} |
| Block 26 | {135, 253, 245, 154, 631, 623, 642, 614} |
| Block 30 | {125, 245, 354, 153, 621, 642, 634, 613} |

A test of the blocks against the vertices of cube 2 shows that they are all consistent. This *could* be a solution for our set. But it is possible to have eight blocks that are consistent with a cube yet still can't be arranged to form the cube. For example, if there are three blocks that fill the same two vertices, but don't fill any others, then there would be only five blocks left to complete the other six vertices in the cube.

This can be represented by a bipartite graph. The eight vertices of the cube $V_1$ and the eight blocks in the set $V_2$ form the vertices of the graph. The edges of the graph are defined by connecting a block in set $V_2$ to the vertices that is shares in set $V_1$. Determining whether this particular cube is a solution is the same as finding a complete matching in the bipartite graph. The requirements for the existence of a solution are given in Hall's Marriage Theorem.




However, determining an actual matching is not a simple matter. Fortunately, the nature of the blocks allows us to simplify this considerably. The 30 vertices of the blocks are identical to the 30 vertices of the solution cubes. Furthermore, there are only three possible ways the vertices of one block can compare with those of a cube.

1. It could have no vertices in common. If this is the case, as we saw in example one, then a solution with that particular cube is not possible.
2. It could have exactly two vertices in common*.
3. It could be identical to the solution cube; in which case it would have all eight vertices in common.

*Consider a block that contains a vertex in common with a particular solution cube. The other three sides can be rearranged in 6 possible ways. In the initial position, all 8 vertices match. If we swap either of the two sides, the unchanged side will yield a second common vertex. if we try to interchange all three sides, the result must be a rotation of the vertex, leaving that vertex itself as the second common vertex.

We proceed with the solution in two ways – using a matrix or a graph.

**Using a Matrix**

We can create an $8 \times 8$ matrix, with each column representing one of the vertices of the cube under consideration – cube 2 in this case.

Each row will correspond to one of our eight blocks.  Entries will be 1 if the vertex of cube 2 is shared with the block of that row, and zero if it does not.

| | |
|---|---|
| Cube 2 | {124, 143, 135, 152, 642, 634, 653, 625} |
| Block 3 | {125, 154, <u>143</u>, 132, 652, 645, <u>634</u>, 623} |

Block 3 shares the second and the sixth vertex of cube 2, so the first row of our matrix will be

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The full matrix for cube 2, as a possible solution for our set of blocks will be:

$$
\begin{array}{l}
\text{block } 3 \\
\text{block } 9 \\
\text{block } 12 \\
\text{block } 14 \\
\text{block } 17 \\
\text{block } 22 \\
\text{block } 26 \\
\text{block } 30
\end{array}
\quad
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
$$

If cube 2 *is* a solution to this set, then each vertex (column) of the cube must be filled by a distinct block (row) in our set. This amounts to being able to rearrange this matrix – through row swaps only – so that it has a diagonal of all ones.

This matrix can be rearranged into:

$$
\begin{array}{l}
\text{block } 12 \\
\text{block } 3 \\
\text{block } 14 \\
\text{block } 9 \\
\text{block } 26 \\
\text{block } 30 \\
\text{block } 22 \\
\text{block } 17
\end{array}
\quad
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

This means that block 12 of our set can fill the first vertex of cube 2, block 3 of our set can fill the second vertex of cube 2, and so on. We can see that cube 2 *is* a solution to this set.

For any cube *i* and any set of 8 blocks, each row of such a matrix will consist of:

1. All 1s – if the block is identical to cube *i.*
2. Two 1s and six 0s if the block is consistent but not identical
3. All 0s – if the block is inconsistent with the cube.

Clearly, if there is a row of zeros, cube *i* is not a solution to the set.

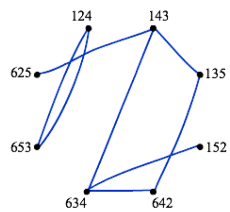
However, it is possible for a matrix to have no all-zero rows and still not have an arrangement with all diagonal elements equal to 1.

I have a program to determine whether a matrix can be rearranged with a diagonal of all 1s, but this is cumbersome to impossible to do by examination.

**Using a Graph**

Another way to assign blocks to vertices of the solution is to create a planar graph in which the eight vertices of the solution cube are the vertices of the graph. The edges are determined, one for each block, by connecting the two vertices that the given block shares with the cube. If a block is identical to the cube, then any *single* edge is drawn, representing two possible vertices of the solution that the block can fill.

Going back to our list of vertices in cube 2 and the set of blocks and *their* vertices, we can draw the following graph.  Block 3 gives the edge connection 143 and 634, block 9 gives the edge between 152 and 634, and so on.




In such a graph, a *connected set* is a set of vertices in which each vertex is connected to every other in the set by some path, and to no vertices that are not part of the set.

Since each vertex of the cube must be filled by one of the blocks (edges), in order for a graph to represent a solution, every connected set of *n* vertices must include at least *n* edges.  Because each block is represented by at most one edge, every connected set of *n* vertices must include *exactly n* edges. It is easy to determine visually when a graph represents a solution, since every set of *n* vertices and *n* edges must contain a circuit. If it has a connected subgroup with fewer edges than vertices, it cannot have a circuit, and hence does not represent a solution cube for that set of blocks.

To illustrate, these graphs would show that the cubes being analyzed are *not* solutions:

In our graph we have two connected sets

{124, 635} and {625, 143, 135, 152, 634, 642}

each having the same number of edges as vertices, and each containing a circuit. This means that cube 2 is a solution for this set of blocks.

The graph can then be followed to determine exactly *how* to arrange the blocks into the solution. I have a couple of different solver programs in which the user can input a set of 8 blocks and get instructions for where to place each block and how to orient it, but determining which solution cube to aim for is the biggest hurdle.

**Sadisticube Variations**

The standard Sadisticube, which we have analyzed here, has each block with six different colors painted on its six different faces, and all blocks in the set have the same six colors. Analysis beyond simply determining a solution has yielded some interesting results and can be found in subsequent papers. Any given block is consistent with 21 solution arrangements. Certain sets have improper solutions, that is solutions with only five or four, or even three colors on their faces. A process for hand-solving the cube can be done using sets of compatible vertices.

In addition to the standard Sadisticube, there are other variations available: Sadisticube 7, where each block is colored with 6 of 7 possible colors. Sadisticube 5 which have blocks with more than face colored the same and only 5 total colors. Sadisticube Touch, which has different textures on each face and can be done purely by feel. There are other extensions and combinations of these as well. The possibilities are many and the analysis continues to be quite interesting.